The Steady State Behavior of Randomly Perturbed Dynamical Systems Near Stable Equilibria

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Abstract

in this paper, wc investigate the statistical behavior of nonlinear systems near stable equilibria in the presence of additive white noise. In the case of certain carrier synchronization loops, such as those that track a combination of suppressed and residual carrier modulations or QAM modulation, it is known that the phase in the steady-state may lock at onc of several possible states, which in general are not equiprobable. To investigate the behavior of such nonlinear stochastic systems in a neighborhood of a stable equilibrium point, wc consider a random realization of the state near an equilibrium point (in the form of a stochastic observation equation) and wc analyze the corresponding Zakai equation, whose solution is the probability density function conditioned on the observation equation. We numerically solve a Zakai equation corresponding to a hybrid loop previously proposed to track a combined suppressed/residual carrier signal. Motivated by questions regarding the performance of phase-locked loops in the stcmdy-state, wc consider the existence of a time-independent equation, whose solution is the steady-state probability density function (PDF) of the system conditioned on the state (equilibrium point) which the system is locked on. Our approach is based on a deterministic approximation to the behavior of a particular Zakai equation in the steady-state. In particular, wc show that asymptotic of the Zakai equation (given that the state is continuously locked) can be approximated by an infinite dimensional eigenvalue problem, where the eigenvalue is a random perturbation of zero and the eigenfunction is the approximate PDF conditioned on the locked state. A detailed analysis of this eigenvalue problem and a numerical procedure for approximating the desired conditional PDF are shown. Numerical results for the hybrid loop indicate that the solutions to the eigenvalue problem agree with those obtained by numerical integration of the Zakai equation.

1 Introduction

Carrier synchronization loops are often employed in coherent communications to provide a phase-lockccl reference in the receiver. Some of the loops, such as the onc described in [1] to demodulate Quadrature Amplituded Shift Keyed (QASK) signals, exhibit multiple stable lock points. Another example is the hybrid carrier and modulation tracking loop which is used to trace a combination of suppressed and residual carrier modulation [2]. The latter estimates the incoming phase using a weighted sum of the estimates provided by a Phase Locked loop (1'1,1,) and a Costas loop to enhance the overall performance and take advantage of the total received power, be it in the tone or the data sidebands. In the analysis of such loops, it is often assumed that the loop locks at zero phase error and no consideration is given to the other lock points. The primary reason being that the theory to tackle the analysis of stochastic systems with multiple lock points is not mature to the point where the conditional phase error density (conditioned on a specific lock point) can be derived. The Fokker-Planck equation (FPE) provides only the combined density but not the conditional densities, which arc needed for a detailed analysis of the probability of bit error and the mean-squared phase error performance (phase filter) taking into account the various lock points and their respective probabilities.

The fundamental issue to be addressed in this paper is as follows: Given a stochastic differential equation (SDE) whose deterministic part has multiple asymptotically stable equilibria, dots there exist a decomposition of the solution to its associated FPE into a weighted sum of conditional probability density functions (PDF), where each PDF is conditioned on the event that the state is in a neighborhood of one of the equilibrium points? If this decompostion is possible, then the weights represent the probability that the solution to the SDE in steady state is in a neighborhood of the corresponding equilibrium point. Our approach to this general question is based on numerical integration of the Zakai equation [3]-[4]. Given a set of observations of the state, the Zakai equation provides us with the PDF of the state conditioned on the observations. At least numerically, the answer to the above question is yes. in the case of certain carrier synchronization loops such as those mentioned above, it is known that the phase in steady-state will lock at one of several possible states. Motivated by questions regarding the performance of the loop in steady-state, we will consider the existence of a time-independent equation, whose solution is the PDF of the phase in steady-state conditioned on the state which the phase is locked on. Our approach to the second question is based on a deterministic approximation to a particular Zakai equation in steady-state.

The remainder of this paper proceeds as follows. In section 2, we summarize some basic results on the Zakai equation and its relation to the Forward Kolmogorov Equation (FKE). We will also discuss a numerical approximation scheme for the Zakai equation which will be used in the subsequent. numerical results. The theory is then applied to the previously mentioned hybrid loop, which has two stable equilibrium points. We will construct a sequence of observations of the phase while it is locked in each of the equilibrium positions. Using each sequence of observations separately, we numerically integrate the Zakai equation to obtain two PDFs which are conditioned on the locked state. We then show that the numerical solution to the FPE (the steady state FKE) decomposes into a weighted average of the two conditional PDFs. In section 3, We will show that the asymptotic of the Zakai equation (given that the state is continuously locked) can be approximated by an infinite dimensional eigenvalue problem, where the eigenvalue is a random perturbation of zero and the eigenfunction is the approximate PDF conditioned on the locked state. A numerical procedure for approximating the desired conditional PDF is shown along with numerical results for the hybrid loop.

2 Basic introduction to the Zakai equation

In this paper we are primarily interested in stochastic differential equations of the form

$$\dot{\phi}(t) = f(\phi) + n(t)\ , \quad \phi(0) = \phi_0, \tag{1}$$

where $\phi(t)$ is a vector in R^d, (d-dimensional Euclidean space), $f(\cdot)$ is a bounded continuous function from R^d into R^d, and $n(t)$ is a white noise vector in R^d with power spectral matrix Q. The initial condition ϕ_0 is assumed to have a known PDF $P_0(\phi)$. It is well-known that the PDF $P(\phi, t)$ of the process $\phi(t)$ is the unique solution to the classical FKE [5]

$$\frac{\partial}{\partial t} P(\phi, t) = LP(\phi, t), \tag{2}$$

where L is the Fokker-Planck operator given by

$$LP(\phi,t) = \sum_{i=1}^{d}\sum_{j=1}^{d}\frac{q_{ij}}{2}\frac{\partial^2}{\partial\phi_i\partial\phi_j}P(\phi,t) - \sum_{i=1}^{d}\frac{\partial}{\partial\phi_i}\left(f_i(\phi)P(\phi,t)\right), \tag{3}$$

where q_{ij} is the (i,j) element of the matrix QQ^T, and $f_i(\phi)$ is the i^{th} component of the vector $f(\phi)$. Equation (2) is posed with the condition that the solution be integrable on R^d. Suppose now that the process $\phi(t)$ is being observed. Let the observation process $z(t)$ be given by

$$z(t) = h(\phi) + v(t), \tag{4}$$

where $h(\cdot)$ and its derivatives are bounded continuous functions from R^d to R^k, and $v(t)$ is a R^k-valued white noise process independent of $n(t)$ and ϕ_0. Without loss of generality, we may assume that $v(t)$ has unit power spectral density. We seek a conditional PDF for the process $\phi(t)$ conditioned on the observation process $\{z(s) : 0 < s \leq t\}$. This conditional PDF is the unique solution of the Zakai equation, which is formally stated below. A detailed treatment of the Zakai equation can be found in [3]-[4]. Intuitively, we expect the Zakai equation to be a stochastic generalization of the FKE since it depends on the random observation process $z(t)$. in fact, the Zakai equation is a linear stochastic partial differential equation (PDE) with multiplicative noise. Solvability theory for such equations cannot be treated within the framework of the deterministic theory of PDE's. Rather, proper and rigorous treatment of the Zakai equation must be done within the context of Ito Calculus [5]- [7]. Some basic results from Ito Calculus are presented in Appendix A.

Following the treatment in Appendix A, let $B(t)$ be a Brownian motion process. We rewrite the state equation (1) and observation equation (4) in Ito form (see Appendix A)

$$d\phi(t) = f(\phi)dt + dB(t)\ , \qquad dY(t) = h(\phi)dt + dV(t), \tag{5}$$

where $dY(t) = z(t)\,dt$, and $V(t)$ is a Brownian motion process independent of $B(t)$ and ϕ_0. For notational simplicity only, we assume that both $B(t)$ and $V(t)$ have unit power spectral matrices. The unnormalized conditional PDF of $\phi(t)$ conditioned on the observation process $Y(t)$ up to time t, which will be denoted simply as $P(\phi,t)$, is the unique solution to the Zakai equation

$$dP(\phi,t) = LP(\phi,t)dt + P(\phi,t)h^T(\phi)dY(t), \qquad P(\phi,0) = P_0(\phi). \tag{6}$$

A derivation of the Zakai equation is sketched in Appendix B. The Zakai equation is typically integrated numerically with possibly discrete observations. Here, we will briefly mention the Splitting-up method, which is explained in [4]. We assume a discrete observation process of the form

$$z(n) = h(\phi(n)) + v(n), \tag{7}$$

where $v(n)$ is white noise with unit power spectral matrix. First, we use a uniformly spaced finite-diffcmncc scheme on a finite grid to discretize the spatial domain. We obtain a system of stochastic differential equations of the form

$$dP(t) = L_h P(t) dt + B_h P(t) dY(t), \tag{8}$$

where $P(t) = (P^1(t), \ldots P^N(t))$ with $P^i(t)$ denoting the conditional PDF at the grid point ϕ_i, L_h is the matrix representation of the discretized operator L, and B_h is a diagonal matrix with $h(\phi_i)$ in its diagonal. The splitting-up algorithm (using an Euler backwards scheme) for the numerical integration of the finite dimensional Zakai equation (8) is given by

$$(I - \Delta t\ L_h) P_{n+1} = \Psi_{n+1} P_n \tag{9}$$

where $P(n) = P(t_n)$, and Ψ_{n+1} is a diagonal matrix with elements

$$\psi_i = \exp\left(\frac{\Delta t}{2}(z(n+1) + z(n)) h(\phi_i) - \frac{\Delta t}{2} h^2(\phi_i)\right), \tag{10}$$

and Δt is the step size for the Euler scheme. Equations *(9)- (1 O)* arc similar to the genera] solution of a one-dimensional I to equation (sec Appendix A). For a hybrid loop [2], $f(\phi)$ takes the form $f(\phi) = -\alpha \sin(\phi) - \beta \sin(2\phi)$. We used the Splitting-up scheme to integrate the Zakai equation (6) corresponding to this loop, which is governed by the SDE

$$d\phi(t) = -\alpha \sin(\phi)\ dt - \beta \sin(2\phi)\ dt + dB(t),$$

with the discrete observation process

$$z(n) = \cos(\phi(n)) + v(n).$$

This problem was solved on the interval [--n, π] with periodic boundary conditions and with α = 0.2, and β = 1.0. We used spatial resolution Ah $= \pi/16$ and temporal resolution

At = 0.01. The hybrid loop has two points at which the phase locks (i.e., the system has two asymptotically stable equilibria), $\phi = 0$ and $\phi = \pi$. There is another unstable equilibrium point between the two stable ones whose location is determined by α and β. To simulate the process of phase locking, we initialized the system (and thus the observation process) near the equilibria. The initial distribution for the Zakai equation was centered near $\pi/2$. in Fig.1, we show the solutions of the Zakai equation at $t = 5000$ (i.e., in steady-state) conditioned on the locked points. Also in Fig.], we show the solution to the FKE for the same initial conditions and same time. Again, the solution to the FKE reaches a steady-state. Using a least squares method, we solved for two constants c_1 and c_2 such that

$$P(\phi) = c_1 P_0(\phi) + c_2 P_\pi(\phi), \tag{11}$$

where $P(\phi)$, $P_0(\phi)$, and $P_\pi(\phi)$ are the steady state numerical solutions to the FKE, the Zakai equation locked at $\phi = 0$, and the Zakai equation locked at ϕ = n- respectively. By Bayes theorem, c_1 and c_2 are the probabilities that the phase in steady state locks at $\phi = 0$ and $\phi = \pi$ respectively. Using the least squares values for c_1 and C_2, we formed $\hat{P}(\phi) = c_1 P_0(\Phi) + c_2 P_\pi(\phi)$. $\hat{P}(\phi)$ is also shown in Fig.]. Comparing $\hat{P}(\phi)$ with $P(\phi)$, we can see that the solution to the FKE decomposes into a weighted sum of the two conditional PDFs obtained by solving the Zakai equation. in principle, we can obtain c_1 and C_2 by evaluating equation (1 1) at any two fixed values of ϕ. However, since the conditional PDFs were obtained numerically, we used a least squares solution, which seems to be very accurate.

3 Decomposition of the steady-state PDF into a weighted sum of conditional PDFs

The numerical integration results in the previous section showed that the Zakai equation can track the PLL system as it becomes locked at one of the equilibrium points. We also showed through numerical integration that for a fixed time (large enough to be considered in steady - state) the solution of the FKE cau be decomposed into the weighted sum of two solutions of the Zakai equation. There are two issues which we will now address. First, the solutions to the Zakai equation depended on the observation process, which is one particular realization of the process $\phi(t)$ with additive observation noise. Second, in practical applications, we would like to obtain a steady-state decomposition without having to integrate forward in time. Thus, we seek a decomposition of the FPE similar to that given for the FKE, and we would like to obtain the steady-state conditional PDFs directly by solving a time-independent equation. Our approach will be general. That is, we consider stochastic dynamical systems whose deterministic part has a finite number of asymptotically stable equilibrium states, and we will address the notion of a steady-state conditional PDF, where we condition on the event that the state is in a neighborhood of an equilibrium point.

In the limit as the random perturbations go to zero, the stochastic system behaves like a deterministic one [5]. in this case, the steady-state PDF approaches a sequence of impulses centered at the equilibrium points. When the noise variance becomes relatively large, the solution to the dynamical system jumps from one equilibrium state to another. The corresponding steady-state PDF approaches that of a uniform random variable. Most physical systems lie somewhere in between the two extremes, and in general the solution to the FPE has relative maxima at the equilibrium states corresponding to the deterministic part. in principle, even if the noise variance is very small, the expected transition time from one equilibruim state to another is finite. However, if this expected transition time among equilibrium states is large enough, such as in the case of the hybrid loop, then we may refer to a conditional steady-state PDF, conditioned on the event that in steady-state the system is in a neighborhood of a specific equilibrium point.

For simplicity only, we consider scalar systems. More formally, consider the SDE in R^1 given by

$$d\phi(t) = f(\phi)dt + dB(t), \tag{12}$$

where the Wiener process $dB(t)$ has unit power spectral density. We assume that the corresponding deterministic system has m asymptotically stable equilibrium points $\{e_1, \ldots, e_m\}$. That is, $f(e_k) = 0$ and $f'(e_k) < 0$ for $k = 1, \ldots, m$. Let $P(\phi)$ be the solution to the associated FPE $LP = 0$. In steady-state, we assume that the expected transition time from one neighborhood of a stable equilibrium point to another is large enough such that the notion of a steady-state conditional PDF has physical recalling. In this case, we may define the function $P_k(\phi) = P(\phi | \phi \in N(e_k))$, where $N(e_k)$ is a neighborhood of e_k. As a result, there exist constants $c_1, \ldots. c_m$ such that

$$P(\phi) = \sum_{k=1}^{m} c_k P_k(\phi), \tag{13}$$

where c_k is the probability of being in $N(c_k)$ in steady-state. Our immediate aim is to obtain an approximate elliptic partial differential equation for $P_k(\phi)$. To this end, consider the observation process defined by (7). The corresponding difference equation for the conditional PDF (with the dependence on k suppressed) is given by (9).

Let us assume that for some n sufficiently large, we observe the system in $N(c_k)$ (i.e. the phase is locked at the point c_k). 'J'bus, wc may usc the approximation

$$\phi(n) = c_k + \xi(n), \qquad z(n) = h(c_k + \xi(n)) +\!\!- \; v(n),$$

where $\xi(n)$ is **a** random deviation of the state from the locked position, and $v(n)$ is the sequence of measurement noi sc, assumed to bc an idd sequence. in particular, the variance of the measurement noise is assumed to be of the order A (-t) ($o(\Delta t)$) uniformly in n. Expanding the observation process about $\xi(n)$ we obtain

$$Z(?2 + 1) + z(n) = 2h(c_k) + h'(c_k)\Big(\xi(n) + \xi(n+1)\Big) + v(n \text{ -1" } 1) + v(n) + o\big(\xi^2(n)\big)$$

which implies that

$$\frac{\Delta t}{2}(z(n+1) + z(n)) = \text{At } h(q) + \lambda_n, \tag{14}$$

where

$$\lambda_n = \frac{\Delta t}{2}(h'(c_k)(\xi(n) + \xi(n+1)) + v(n + \text{I}) + v(n)) \text{ -t- } o(\Delta t \xi(n)^2),$$

In the ensuing analysis we will be interested in the asymptotic. behavior of the sequence λ_n. In particular, wc assume that λ_n satisfies

$$\lim_{n\to\infty} \text{Var}\,(\lambda_n) = o\,,$$

where Var$(\cdot)$ denotes variance, and σ is a constant, which is $o(\Delta t)$. To this end, we note that λ_n depends on two random variables, $\xi(n)$ and v(n). The i id measurement noise sequence $v(n)$ is assumed to have a corresponding bounded sequence of variances, where the bound is uniform in n. The other random sequence is $\xi(n)$, which is a small deviation from an equilibrium point. If we consider $\xi(n)$ as a sampling at time t_n of the continuous process $\xi(t)$, and if we assume that being locked at an equilibrium point c_k in steady-state implies that $E(\xi^r(t))$ is negligible for some $r > 0$ and for arbitrarily large t, and if we assume that f is r-1 times differentiable, then $\xi(t)$ satisfies the SDE

$$d\xi(t) = \sum_{m=0}^{r-1} \frac{f^{(m)}(c_k)}{m!} \xi^m(t)dt + dB(t),$$

where $f^{(m)}(\cdot)$ denotes the m^{th} derivative of $f(\cdot)$. The steady-state PDF of $\xi(t)$ exists and can be obtained by solving the associated FPE. We denote this distribution by $g^*(\phi)$. Thus, if $g(\phi, t)$ is the PDF of $\xi(t)$, then

$$\lim_{t \to \infty} \phi^k g(\phi, t) = \phi^k g^*(\phi), \text{ for all } \phi \in \mathrm{R}, \text{ and } k = 1, 2.$$

(recall that $g(\phi, t)$ and $g^*(\phi)$ have compact support on R). Thus, By the Dominated Convergence Theorem [8], we have

$$\lim_{t \to \infty} \int_R \phi^k g(\phi, t) = \int_R \phi^k g^*(\phi), \quad k = 1, 2,$$

which implies that the sequence of variances of $\xi(t)$ convergences to a finite constant as t approches infinity. As a result, if for n arbitrarily large the system remains locked at the equilibrium point c_k, then λ_n (in steady state) converges to a random variable λ with variance $o(\Delta t)$. Note that other arguments can be made which do not require any smoothness on the function f.

Substituting (14) into (10) and expanding the exponential term we obtain

$$\psi_i = \exp\left(\Delta t\ h(c_k)h(\phi_i) - \frac{\Delta t}{2} h^2(\phi_i)\right) (I + o(\Delta t)).$$

As a result, we approximate the difference equation for P_n by

$$(I - \Delta t\ L_h)\, P_{n+1} = A P_n + \lambda_n A P_n, \tag{15}$$

where A is a diagonal matrix with diagonal entries

$$\exp\left(\Delta t\ h(c_k)h(\phi_i) - \frac{\Delta t}{2} h^2(\phi_i)\right)$$

By letting n approach infinity in equation (1 5), we see that the existence of a steady state conditional PDF is equivalent to the statement that the following equation has a solution:

$$(I - \Delta t\, L_h)\, P = (A + \lambda A) P,$$

where λ is a random variable with $o(\Delta t)$ variance. From this we finally obtain the steady-state conditional PDF

$$A^{-1}\ (I - \Delta t\, L_h - A)\, P = \lambda P. \tag{16}$$

in principal, the eigenvalue λ corresponding to the eigenvector P is not known, except we do know it has a variance which is $o(\Delta t)$. Fox the PLL example of section 2, we examined the eigenvalues of the matrix $A^{-1}(I - \Delta t\, L_h - A)$ for different choices of α and β and for Δt in the range $0.0001 \leq \Delta t \leq 1.0$. We found that the magnitude of the smallest eigenvalue is always $o(\Delta t)$. We also found that this eigenvalue is always negative. In fact, eigenvectors corresponding to positive $o(\Delta t)$ eigenvalues sometimes had negative components. Also, in our numerical experiments, we found that the numerical solutions to equation (16) were not sensitive to Δt. For the purpose of analysis, we may set $\Delta t = 1$, even though in general Δt may play an important role in the numerical solutions. The steady-state equation becomes

$$\left(A^{-1} - A^{-1} L_h - I\right) P = \lambda P, \tag{17}$$

where $\Delta t = 1$ in the diagonal components of A.

in Fig.2, we show the normalized solutions to equation (17) for $h(\phi) = \cos(\phi)$, and λ set to the negative eigenvalue with smallest magnitude. The results are compared to the solutions to the Zakai equation which were obtained in section 2. The results suggest that the resulting eigenvalue problem is a good steady-state approximation to the condtional PDF of the phase $\phi(t)$, conditioned on the state at which the phase is locked. This provides us with a possible way to compute P. The fact that we chose λ to be the negative eigenvalue with smallest magnitude can be justified in the following way. First, we know that its variance is $o(\Delta t)$. The fact that it should be negative can be explained by examining the infinite dimensional analogue of equation (1 7), which is given by the elliptic partial differential equation

$$g(\phi, e_k) L P(\phi) + (1 + \lambda - g(\phi, e_k))\, P(\phi) = 0, \tag{18}$$

where L is the Fokker-Planck operator defined in equation (3), and

$$g(\phi, c_k) = \exp\left(\frac{1}{2}h^2(\phi) - h(c_k)h(\phi)\right). \tag{19}$$

Equation (18) can be posed on a periodic domain or on a large but finite domain with zero boundary conditions (i.e., since the solution is a PDF, it can be approximated by a function with compact support). It is well-known from the theory of partial differential equations that a unique solution to equation (18) exists if zero is an eigenvalue of the elliptic operator $D = g(\phi)L + (1 + \lambda - g(\phi))$. On the other hand, we may also view the solution of (18) as the asymptotic limit of the solution to the equation $\dot{P}(t, \phi) = DP(t, \phi)$. For the latter equation to have a classical solution in steady-state, it is required that the rate of change of the energy $\frac{d}{dt}|P(t)|^2_{L_2(\Omega)}$ be nonpositive. This is equivalent to to the condition $< DP, P>_{L^2(\Omega)} \leq 0$ for all P in $C_0^2(\Omega)$ (twice-differentiable functions with compact support on a bounded domain Ω), where $< \cdot, \cdot >_{L^2(\Omega)}$ denotes the L^2 innerproduct over Ω. Note that

$$< DP, P >_{L^2(\Omega)} = < g(\phi, c_k)LP, P >_{L^2(\Omega)} + < (\lambda + 1 - g(\phi, c_k))P, P >_{L^2(\Omega)}.$$

Since $g(\phi, c_k)$ is nonnegative and the Fokker-Planck operator is dissipative (otherwise the solution of $\dot{P}(t) = LP(t)$ would not have a finite limit in steady-state), then a sufficient condition for the dissipitivity of the operator D is that $\lambda + 1 - g(\phi, c_k)$ be nonpositive. To illustrate this condition, consider the example of the hybrid loop of section 2. A plot of $g(\phi, c_k)$ with $h(\phi) = \cos(\phi)$ is shown in Fig.3. We can see that $1 - g(\phi, c_k)$ can be slightly positive for both equilibrium points $c_1 = 0$ and $c_2 = \pi$. If λ is, say, less than –0.5, then D is guaranteed to be a dissipative operator. This is only a sufficient condition, but it illustrates the problem that can occur if λ is positive.

Although our analysis has been in the context of the hybrid loop example of section 2 with $h(\phi) = \cos(\phi)$, the same results can be generalized to higher dimensions and to any bounded function $h(\phi)$. The choice for $h(\phi)$ only affects the unnormalized PDF. By examining $g(\phi, c_k)$, we see that this is a parabolic function of $h(\phi)$ which attains one minimum at $h(c_k)$ and grows exponentially as ϕ moves away from c_k (see Fig.3). As a result, we can make several more conclusions regarding equation (18), whose solution we will now call the (unnormalized) steady-state conditional PDF (conditioned on the equilibruim state). In particular, equation

(18) can be viewed as a generalization of the Fokker-Planck equation. The first term in equation (18) is given by

$$g(\phi, c_k)LP(\phi) = \sum_{i=1}^{d}\sum_{j=1}^{d}\frac{g(\phi, c_k)q_{ij}}{2}\frac{\partial^2}{\partial\phi_i\partial\phi_j}P(\phi) - \sum_{i=1}^{d} g(\phi, c_k)\frac{\partial}{\partial\phi_i}\left(f_i(\phi)P(\phi)\right).$$

Since $g(c_k, c_k) << 1$ and $g(\phi, c_k) > 1$ for ϕ away from c_k, the diffusion coefficient (coefficient of the Laplacian), which is proportional to the power spectral matrix of the stochastic perturbation, is attenuated near c_k and magnified away from c_k. Expanding the second part of $g(\phi, c_k)LP(\phi)$, we obtain

$$\sum_{i=1}^{m}\left(g(\phi, c_k)\frac{\partial f_i}{\partial\phi_i}(\phi)P(\phi) + g(\phi, c_k)f_i(\phi)\frac{\partial P}{\partial\phi_i}(\phi)\right) = (g \text{ div } f)P + (gf)\cdot\nabla P. \quad (20)$$

The coefficient of $P(\phi)$, which is g div f, is called the growth coefficient. If it is positive, then it shifts the real part of the eigenvalues of the diffusion to the right (indicating growth in the solutions) and if it is negative if shifts the real part of the diffusion eigenvalues to the left, which introduces damping to the solutions. Near the stable equilibria, the Jacobian of f must have negative eigenvalues, which implies that div f <0 (the trace of the Jacobian of f is equal to div f). Thus, multiplying div f by $g(\phi, c_k)$ will magnify the damping of the solutions away from c_k and attenuate the damping near c_k. The advection coefficient (i.e. the coefficient of ∇P) affects the rate at which initial conditions in an evolution equation are propagated. Its relative effect on the solutions near the stable equilibria is secondary since f is close to zero in these regions, In general, multiplying the Fokker-Planck operator by $g(\phi, c_k)$ tends to make the corresponding solutions exhibit narrower peaks near c_k and be more flat away from c_k. In particular, the relative maxima near other equilibrium points in the solutions of the FPE become flatter and closer to zero. The dissipation of the solution near the other equilibria is magnified due to the second term in equation (18). This term is

$$(\lambda + 1 - g(\phi, c_k))P,$$

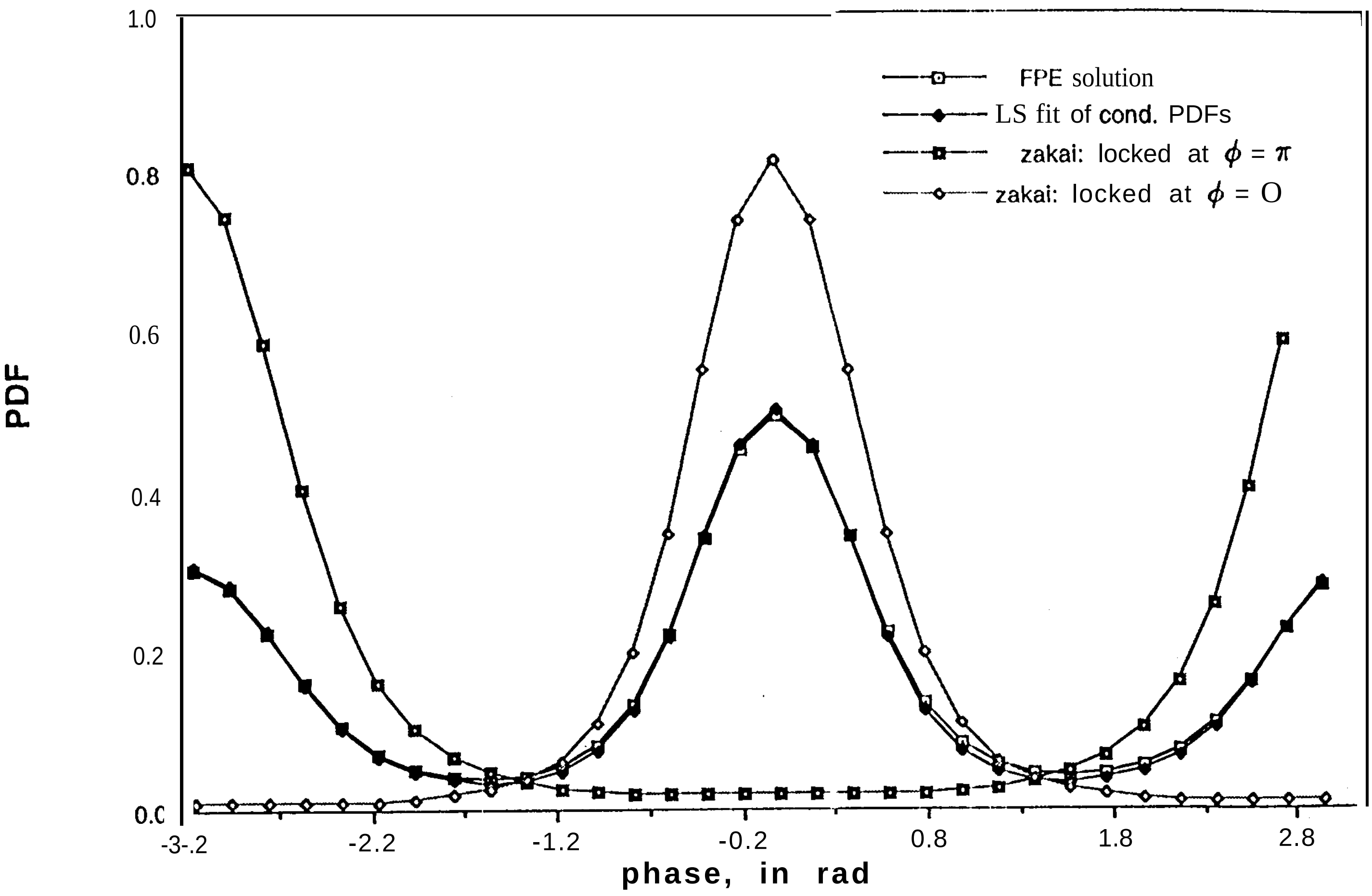
which essentially acts as another dissipation term, with the dissipation magnified away from c_k (where $g(\phi, c_k)$ is maximum) and negligible near c_k. Again, solutions to equation (18) exhibit further decay as we move away from c_k, particularly near the other equilibria. Thus, the qualitative form of equation (18) matches the physical behavior which one expects from a conditional PDF. In particular, numerical solutions to equation (18) agree with those of the Zakai equation. However, equation (18) was derived from asymptotic analysis of a particular approximation scheme of the Zakai equation in steady-stalm. However, if wc had used a forward Euler scheme to discretize the Zakai equation, our asymptotic analysis would have resulted in equation (18) as well. Wc expect that any spatial discretization scheme for the Zakai equation would yield an approximation to (18) in the limit as time approaches infinity.

Finally, once wc have obtained $P_k(\phi)$ by solving equation (1 8) using the numerical algorithm in this section, wc may solve for the coefficients c_k by solving the FPE for $P(\phi)$ and by evaluating equation (13) at m distinct points ϕ_i. If wc combine the results of Fig.1 and Fig.2, wc can conclude that the constants c_k obtained numerically using the solutions to the Zakai equation (scc section 2) would be the same obtained using instead the the solutions to (18).

4 Conclusion

In this paper wc investigated the behavior of nonlinear stochastic systems in a neighborhood of a stable equilibrium point. Wc derived a time-independent equation for the conditional PDF of the state conditioned on a given stable equilibrium point,. Wc further established the methodology using a hybrid synchronization loop. The results in this paper continue to hold in the case of multiplicative noise with only minor modifications to the Fokker-Planck equation. in particular, the asymptotic analysis of the Zakai equation remains essentially unch angcd.

Figure 1 - Steady state and Conditional PDFs



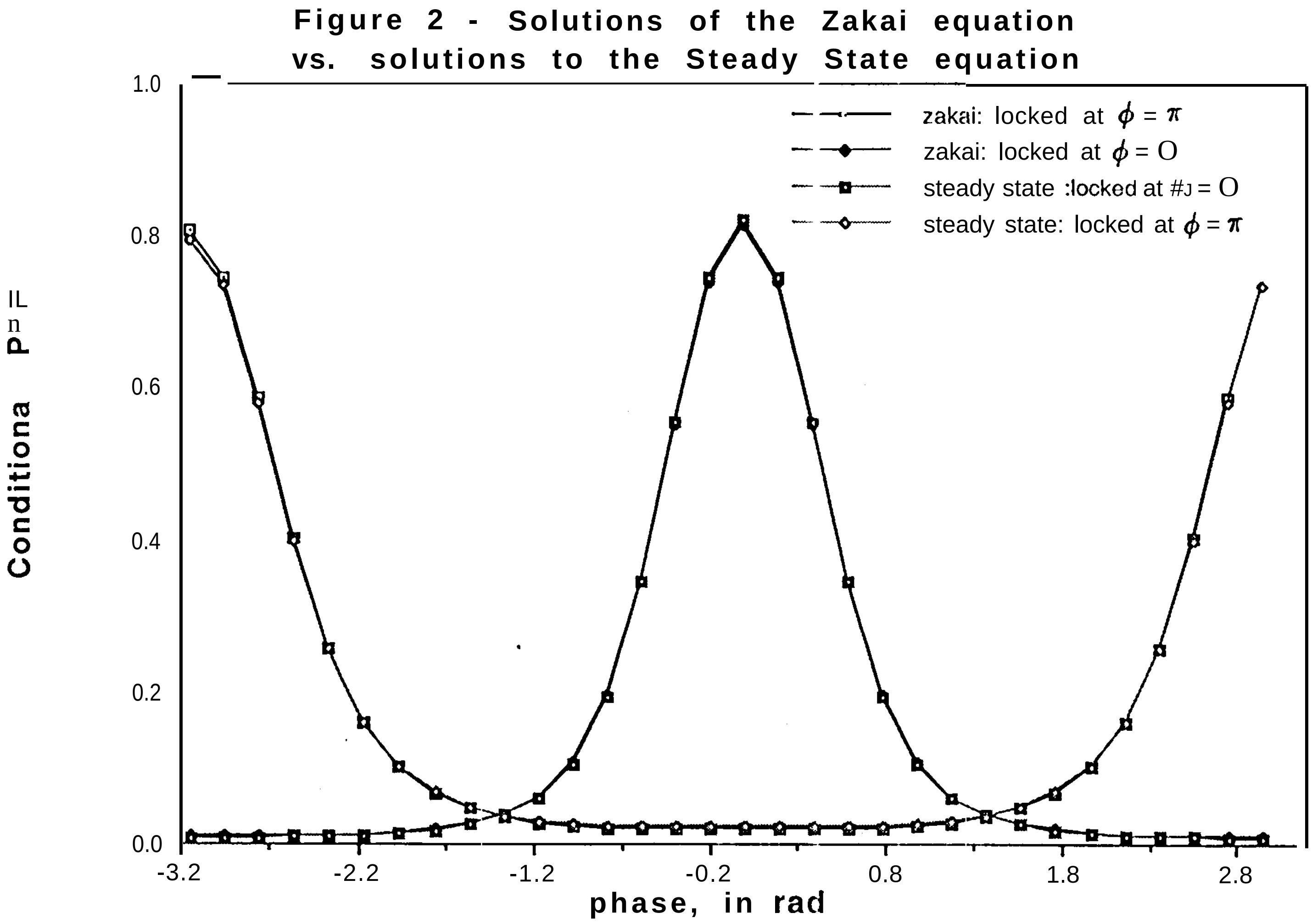
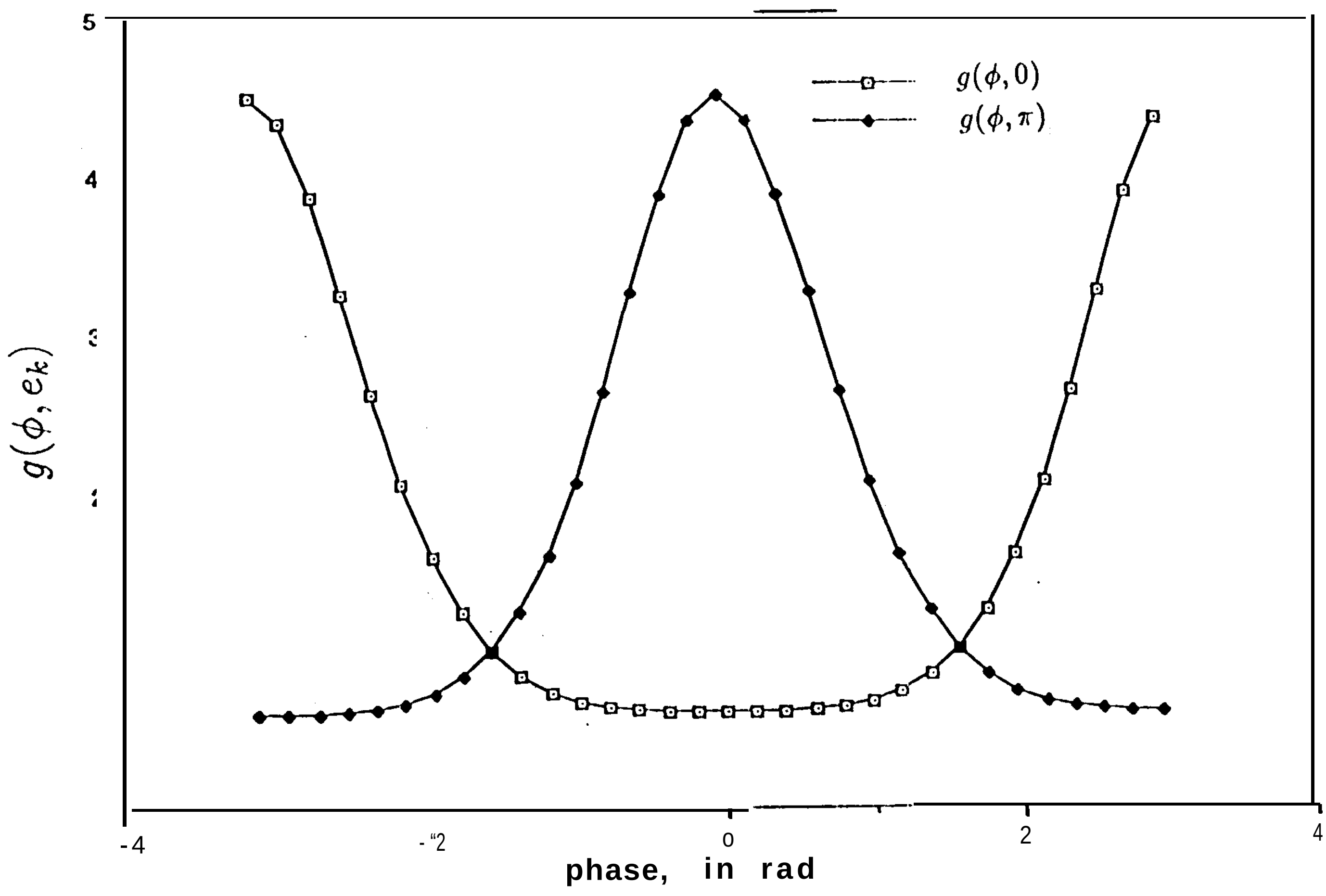


Figure 2 - Solutions of the Zakai equation
vs. solutions to the Steady State equation
zakai: locked at $\phi = \pi$
zakai: locked at $\phi = 0$
steady state :locked at #J = O
steady state: locked at $\phi = \pi$
Conditiona P n IL
phase, in rad
1.0
0.8
0.6
0.4
0.2
0.0
-3.2
-2.2
-1.2
-0.2
0.8
1.8
2.8

Figure 3 - $g(\phi, e_k)$ **as a function of** phase ϕ



Appendix A
Basic results from Ito Calculus

Wc begin with a formal definition of a probability spat.c. Consider an experiment E whose outcomes are elements of a sample space Ω. Let $\mathcal{F}$ be a sigma-field (set of all possible sets) of observable events, and let $\mathcal{P}$ be a probability measure defined on the sets of $\mathcal{F}$. Wc can think of $\mathcal{P}$ as an abstract representation of a distribution function F characterizing the experiment E. The probability space associated with the experiment E will be denoted by $(\Omega, \mathcal{F}, \mathcal{P})$. For the same experiment E, wc may define another probability space, $(\Omega, \mathcal{F}, \mathcal{P}^+)$, which is related to the original space by the relation $dF(x) = \rho(x)dF^+(x)$, where F^+ is the distribution function associated with $\mathcal{P}^+$, and ρ is called the Radon-Nykodym derivative of $\mathcal{P}$ with respect to $\mathcal{P}^+$. Roughly speaking, switching from one probability space to another amounts to making a change of variables. For example, for any set in $A \in \mathcal{F}$, *wc* have [Chung]

$$\mathcal{P}(A) = \int_A dF(x) = \int_A \rho(x)dF^+(x), \qquad \mathcal{P}^+(A) = \int_A dF^+(x) = \int_A \rho^{-1}(x)dF(x).$$

]n order to define a stochastic differential equation, we would typically need a time-dependent probability space of the form $(\Omega, \mathcal{F}_t, \mathcal{P})$. For example, let $B(t)$ be a Brownian motion such that $dB(t) = n(t)\,dt$, where $n(t)$ is a white noise process [ ]. Wc may let $\mathcal{F}_t$ be the sigma-field generated by the trajectory of B(s) up to time t. Symbolically, wc write $\mathcal{F}_t = \sigma\,(B(s) : 0 \leq s \leq t)$. In this case, a function will be called $\mathcal{F}_t$-measurable if it is a function of the trajectory $B(t)$. For such functions, wc define the]to integral as follows: Let $(t_0, t_1, \ldots .\ t_n)$ be a partition of the interval [0, t] where $t/n = t_k - t_{k-1}$. Then the Ito integral of a $\mathcal{F}_t$-measurable function $g(t)$ is defined as

$$\int_0^t g(s)dB(s) = \lim_{n\to\infty} \sum_{k=0}^{n-1} g(t_k)\left(B(t_{k+1}) - B(t_k)\right), \tag{A.]}$$

whenever the summation converges in the mean-square-sense. Wc note here that the limit of the sum depends on the fact that g(s) is sampled at t_k (as opposed to anywhere in the interval $[t_k, t_{k+1}]$). This differs from the usual definition of recall-square convergence, where the limit dots not depend on where g(s) is sampled. For a discussion on this subject, see [

]. For $B(t)$ as above, we may define the stochastic integral equation

$$\phi(t) = \phi_0 + \int_0^t f(\phi(s))ds + \int_0^t g(\phi(s))dB(s), \tag{A.2}$$

where the first integral is defined as a mean-square Riemann Stieltjes integral in the usual sense [ ], and the second integral is an Ito integral in the sense of equation (A.]). Symbolically, the integral equation is writen as a differential] equation in the Ito form

$$d\phi(t) = f(\phi)dt + g(\phi)dB(t), \quad \phi(0) = \phi_0. \tag{A.3}$$

If $f(\phi) = f_0(t)\phi(t)$ (i.e. $f(\phi)$ is linear in ϕ) and $g(\phi) := g_0\phi(t)$ where g_0 is constant, then in one dimension the general solution to (A .2) is given by

$$\phi(t) = \phi_0 \exp\left(\int_0^t f(s)ds + g_0 B(t) \quad -\$ \right). \tag{A.4}$$

An important result in Ito Calculus is Ito's Lemma. in one-dimension, it states that if u is a smooth function of ϕ, then

$$du(\phi(t)) = \frac{\partial u(\phi)}{\partial \phi} d\phi(t) + \frac{g^2(\phi)}{2} \frac{\partial^2 u(\phi)}{\partial \phi^2} dt. \tag{A.5}$$

Equation (A .3) can be generalized to stochastic systems in R^m, where it is often expressed as as

$$du(\phi(t)) = L^* u(\phi(t))dt, \tag{A.6}$$

where L^* is the adjoint of the Fokker-Planck operator with respect to the $L^2(R^m)$ inner-product. An important corollary of Ito's lemma is the product rule for two solutions to Ito differential equations. Let $x(t)$ and $y(t)$ satisfy

$$dx(t) = f_1(x)dt + g_1(x)dB_1(t), \quad dy(t) := f_2(y)dt + g_2(y)dB_2(t),$$

where $B_1(t)$ and $B_2(t)$ are independent Brownian motion processes. Let $z(t) = x(t)y(t)$. Then $z(t)$ satisfies the Ito differential equation

$$dz(t) = x(t)dy(t) + y(t)dx(t).$$

Appendix B
Summary of the Zakai equation

Following the treatment in the Appendix A, let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, let $B(t)$ be a Brownian motion process, and let $\mathcal{F}_t$ be the sigma- -field generated by the process $B(t)$. We assume that $\mathcal{F}_t$ c $\mathcal{F}$ for each t. On (Ω, .?-, T'), we consider the state equation and observation equation in Ito form

$$d\phi(t) = f(\phi)dt + dB(t) \ , \qquad dY(t) = h(\phi)dt + dV(t), \tag{B.1}$$

where $V(t)$ is a Brownian motion process independent of $B(t)$ and ϕ_0. For notational simplicity only, we assume that both $B(t)$ and $V(t)$ have unit covariances. It is assumed that $\mathcal{Y}_t$, the sigma-field generated by the process $Y(t)$ up to time t, is contained in $\mathcal{F}_t$. Our aim is to sketch the derivation of the Zakai equation, whose solution is $P(\phi, t|\mathcal{Y}_t)$, the unnormalized conditional PDF of $\phi(t)$ conditioned on the observation process $Y(t)$ up to time t. We will denote this function simply as $P(\phi, t)$. Thus, for any infinitely differentiable test function $u(\phi)$, we have

$$E\left(u(\phi(t))|\mathcal{Y}_t\right) = \int_{R^d} u(\phi)P(\phi, t)d\phi \ . \tag{B.2}$$

To obtain the desired equation, we begin by introducing a new probability space $(\Omega, \mathcal{F}, \mathcal{P}^+)$ such that under $\mathcal{P}^+$, the state process is statistically unchanged, but $Y(t)$ becomes a Wiener process independent of $B(t)$. The new space is related to the original space by the relation (see Appendix A) $dF(\phi) = \rho(\phi, t)dF^+(\phi)$, where F and F^+ are the associated distribution functions, and $\rho(\phi, t)$ is the Radon-Nikodym derivative of $\mathcal{P}$ with respect to $\mathcal{P}^+$ given by

$$\rho(\phi, t) = \exp\left(\int_0^t h^T(\phi(s))dY(s) - 1/2 \int_0^t |h(\phi(s))|^2 dY(s)\right). \tag{B.3}$$

In particular, for any test function $u(\phi)$, we have the following relation:

$$c\,E^+\left(u(\phi(t))\rho(\phi, t)|\mathcal{Y}_t\right) = E\left(u(\phi(t))|\mathcal{Y}_t\right) = \int_{R^d} u(\phi)P(\phi, t)d\phi, \tag{B.4}$$

where c is a normalization constant. Using Ito's lemma (see Appendix A), it can be shown that $\rho(\phi, t)$ satisfies the equation

$$d\rho(\phi, t) = \rho(\phi, t)h^T(\phi(t))dY(t). \tag{B.5}$$

Also by Ito's Lemma, $u(\phi(t))$ satisfies the stochastic differential equation

$$du(\phi(t)) = L^* u(\phi(t)) + dB(t),$$

where L^* is the adjoint of the Fokker-Planck operator L with respect to the $L^2(R^m)$ innerproduct (see Appendix A). On the space $(\Omega, \mathcal{F}, \mathcal{P}^+), Y(t)$ is a Wiener process. Thus, equation (B.5) is an Ito differential equation for $\rho(\phi, t)$. On the other hand, since $Y(t)$ is independent of $B(t)$, we may apply the product, rule (see Appendix A) to obtain a differential equation for $d(\rho(\phi, t)u(\phi(t))$. This is perhaps one of the most crucial steps in the derivation (see [ ]). It was facilitated by the fact that on $(\Omega, \mathcal{F}, \mathcal{P}^+), Y(t)$ and $B(t)$ are independent Wiener processes. This motivates why we use this probability space instead of the original one. After expressing the differential equation for $d(\rho(\phi, t)u(\phi(t))$ in integral form, we apply $E^+(\cdot|\mathcal{Y}_t)$ to obtain

$$E^+(u(\phi(t))\rho(\phi,t)|\mathcal{Y}_t) = E^+(u(\phi_0)\rho(\phi,0)) + \int_0^t E^+(L^*[u(\phi(s))]\rho(\phi(s),s)|\mathcal{Y}_s)\, ds \tag{B.6}$$

$$+ \int_0^t E^+\left(h^T(u(\phi, s))\rho(\phi(s))|\mathcal{Y}_s\right) dY(s).$$

Using the equality in equation (11.4), we can see that equation (B.6) is equivalent to

$$< P(\phi,t), u > = < P(\phi,0), u > + \int_0^t < P(\phi,s), L^* u > ds \tag{B.7}$$

$$+ \sum_{i=1}^{k} \int_0^t < h_i(\phi)P(\phi,s), u > dY_i(s),$$

where $< \cdot, . >$ denotes the $L^2(R^d)$ innerproduct. Finally, equation (B.7) is the variational form of the Zakai equation

$$dP(\phi,t) = LP(\phi,t)dt + P(\phi,t)h^T(\phi)dY(t), \qquad P(\phi,0) = P_0(\phi), \tag{B.8}$$

where L is the Fokker-Planck operator given in (3) of section 1. Note that if there are no observations, then $Y(t)$ is zero and the Zakai equation reduces to the Forward Kolmogorov equation.

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